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One-point functions of the XXZ model and Ramanujan's ${}_1\Psi_1$ sum

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Abstract. We apply Ramanujan's sum in q-analysis to evaluate the one-point functions of the XXZ model.

The XXZ model is reformulated in Davies *et al* (1993) from the viewpoint of the representation theory of the quantum affine algebra $U_q\widehat{sl_2}$. Subsequently, multipoint correlation functions of the XXZ model are given in Jimbo *et al* (1992) by means of contour integrals. Although the formulae are explicit, it is not easy to extract information from them, even in the case of one-point functions. The aim of this paper is to give a simple evaluation of one-point functions of the XXZ model by using Ramanujan's ${}_1\Psi_1$ sum, a fundamental ingredient in q-analysis.

Throughout this paper, we fix a number q with $-1 < q < 0$ and use the standard notation of q-shifted factorials

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \qquad \text{and} \qquad (a;q)_n = (a;q)_\infty/(aq^n;q)_\infty.$$

Moreover, since products of q-shifted factorials occur so often, we also use the more compact notation

$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty.$$

As special cases of the formulae in Jimbo *et al* (1992), the one-point functions $P_\varepsilon^\varepsilon(z_1; i)$ ($\varepsilon = \pm$ and $i = 0, 1$) can be described as

$$P_+^+(z_1|i) = -\frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty^2}\oint_{|q^2z_1|<|\eta|<|z_1|}\frac{\mathrm{d}\eta}{2\pi\sqrt{-1}(\eta - z_1)}\frac{\sum_{m\in \mathbf{Z}+(i/2)}(z_1/\eta)^{2m}q^{4m^2-2m}}{(q^2\eta/z_1, q^2z_1/\eta; q^2)_\infty}$$

$$= -\frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty^2}\oint_{|q^2|<|\eta|<1}\frac{\mathrm{d}\eta}{2\pi\sqrt{-1}(\eta - 1)}\frac{\sum_{m\in \mathbf{Z}+(i/2)}\eta^{2m}q^{4m^2+2m}}{(q^2\eta, q^2\eta^{-1}; q^2)_\infty}$$

and

$$P_-^-(z_1|i) = \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty^2}\oint_{|z_1|<|\eta|<|q^{-2}z_1|}\frac{\mathrm{d}\eta}{2\pi\sqrt{-1}(\eta - z_1)}\frac{\sum_{m\in \mathbf{Z}+(i/2)}(z_1/\eta)^{2m}q^{4m^2-2m}}{(q^2\eta/z_1, q^2z_1/\eta; q^2)_\infty}$$

$$= \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty^2}\oint_{1<|\eta|<|q^{-2}|}\frac{\mathrm{d}\eta}{2\pi\sqrt{-1}(\eta - 1)}\frac{\sum_{m\in \mathbf{Z}+(i/2)}\eta^{2m}q^{4m^2+2m}}{(q^2\eta, q^2\eta^{-1}; q^2)_\infty}$$

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for $i = 0, 1$, where each integration path is taken in the counterclockwise direction. As it is easily seen that $P_+^+(z_1|i) = P_-^-(z_1|1-i)$, we discuss only $P_-^-(z_1|i)$ $(i = 0, 1)$.

Jacobi's triple product identity for the theta function

$$\sum_{k\in Z}(-\eta)^k q^{k(k+1)/2} = (q\eta, 1/\eta, q; q)_\infty \tag{1}$$

leads to

$$P_-^-(z_1|0) \pm P_-^-(z_1|1) = \frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2} G^\pm$$

with

$$G^\pm = \oint_{|\eta|=1} \frac{\mathrm{d}\eta}{2\pi\sqrt{-1}\eta} \frac{(\mp q\eta, \mp q\eta^{-1}; q^2)_\infty}{(q\eta, q\eta^{-1}; q^2)_\infty} \tag{2}$$

where the integration path is also taken in the counterclockwise direction. Hence we are interested in the evaluation of the integrals $G^\pm$. It is clear that $G^- = 1$. To evaluate the integral G^+, we recall Ramanujan's ${}_1\Psi_1$ sum.

Ramanujan's ${}_1\Psi_1$ sum can be written as

$$\frac{(\alpha t, q/\alpha t, q, \beta/\alpha; q)_\infty}{(t, \beta/\alpha t, \beta, q/\alpha; q)_\infty} = \sum_{-\infty}^{\infty} \frac{(\alpha;q)_n}{(\beta;q)_n} t^n \tag{3}$$

for $|\beta/\alpha| < |t| < 1$. It has several simple proofs. We refer the reader to Andrews and Askey (1978) and Ismail (1977). It is worth while noting that Jacobi's triple product identity (1) can be thought of as a special case of Ramanujan's ${}_1\Psi_1$ sum. See Andrews (1986).

After the base change $q \mapsto q^2$, set $t = q\eta, \alpha = -1, \beta = -q^2$ in (3). Then we have

$$\frac{(-q\eta, -q/\eta; q^2)_\infty}{(q\eta, q/\eta; q^2)_\infty} = \frac{(-q^2;q^2)_\infty^2}{(q^2;q^2)_\infty^2} \sum_{-\infty}^{\infty} \frac{(-1;q^2)_n}{(-q^2;q^2)_n} (q\eta)^n.$$

Applying this equality to (2), we obtain the evaluation

$$G^+ = \frac{(-q^2;q^2)_\infty^2}{(q^2;q^2)_\infty^2} \oint_{|\eta|=1} \frac{\mathrm{d}\eta}{2\pi\sqrt{-1}\eta} \sum_{-\infty}^{\infty} \frac{(-1;q^2)_n}{(-q^2;q^2)_n} (q\eta)^n = \frac{(-q^2;q^2)_\infty^2}{(q^2;q^2)_\infty^2}.$$

Therefore we have obtained the equalities

$$P_-^-(z_1|0) + P_-^-(z_1|1) = 1 \qquad \text{and} \qquad P_-^-(z_1|0) - P_-^-(z_1|1) = \frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2}$$

which are the same as those in Jimbo *et al* (1992). Consequently we get the desired expression for the one-point functions

$$P_-^-(z_1|0) = \frac{1}{2}\left\{1 + \frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2}\right\} \qquad \text{and} \qquad P_-^-(z_1|1) = \frac{1}{2}\left\{1 - \frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2}\right\}.$$

Since $P_+^+(z_1|\varepsilon) = P_-^-(z_1|1-\varepsilon)$, we also have

$$P_+^+(z_1|0) = \frac{1}{2}\left\{1 - \frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2}\right\} \qquad \text{and} \qquad P_+^+(z_1|1) = \frac{1}{2}\left\{1 + \frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2}\right\}.$$

See also Jimbo *et al* (1993).

Finally we should give a remark that integral (2) is the same as (2.3) in Askey (1983). Although Askey's evaluation is the same as ours, it is repeated here for self-containedness.

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